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Original Research Article

Bayesian Estimation of Kumaraswamy Distribution under Different Loss Functions

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Abstract

In this study, the procedures of Bayesian estimation of the shape parameter of the Kumaraswamy distribution under different prior distributions are examined. The shape parameter of the Kumaraswamy distribution is assumed to follow noninformative prior distributions (such as the extension of Jeffrey's prior distribution, Hartigan Prior distribution, and Uniform Prior distribution) and the informative prior distribution (such as the Gamma distribution and Inverted levy distribution) were adopted in this work. We compared the obtained estimates using their mean square errors under different loss functions (such as precautionary loss function, Squared error loss function, and Albayyati's loss function). The results showed that the behaviour of the Bayesian estimations of the shape parameter of the Kumaraswamy distribution under the Squared Error loss function using Inverted levy prior distribution is the best among all the prior distributions considered.

Keywords: Mean Square Error, Kumaraswamy distribution, Jeffrey's prior, Hartigan Prior, Uniform Prior, Gamma distribution, Inverted levy distribution, Loss Function.

1.0 Introduction

Kumaraswamy(1980) proposed a new probability density function for double bounded random processes known as Kumaraswamy distribution. This probability distribution has a relationship with Beta distribution but it possesses a simple closed-form for both probability distribution function and cumulative distribution function. Kumaraswamy (1987) further proposed a sine-power probability density function because some of the well-known probability distributions like Normal, Beta, log-normal and polynomial-transformed-normal density functions do not fit well to hydrological data.

The Kumaraswamy probability density function proposed by Poondi Kumaraswamy in 1980 was a more generalized probability distribution function for lower and upper bounded random variables. The probability density function and the cumulative distribution function for the Kumaraswamy distribution can be expressed as

$$f(x;\alpha,\beta)=\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1} \qquad ; 0<x<1; \alpha>0; \beta>0 \tag{1}$$

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$$F(x) = 1 - (1 - x^{\alpha})^{\beta - 1} \qquad (2)$$

The two parameters α and β are known as non-negative shape parameters for the Kumaraswamy distribution. Nadarajah (2008), the Kumaraswamy distribution is closely related to three-parameter Beta distribution. In this study, we are focusing on estimating the shape parameter β of this distribution.

$$\frac{\alpha}{B(\gamma,\beta)} x^{\gamma\alpha - 1}(1 - x^{\alpha})^{q-1} \qquad 0 \le x \le 1; \alpha, \beta > 0 \qquad (3)$$

When $\alpha = q$ and $\beta = 1$, Equation (1) is regarded as a power function distribution denoted as Kumaraswamy (q,1) . When $\alpha = 1$ and $\beta = q$, Equation (1) is regarded as a distribution of one minus that power function which is denoted as Kumaraswamy (1,q). Also, when $\alpha = 1$ and $\beta = 1$, Equation (1) is said to follow a uniform distribution U(0,1) which is denoted as Kumaraswamy (1,1).

Jones (2009) highlighted some similarities and differences between the Beta distribution and the Kumaraswamy distributions. The Kumaraswamy distribution has similar characteristics with Beta distribution but it's easier to use than the three-parameter Beta distribution especially in a simulation study. Fletcher and Ponnamblam (1996), Sundar and Subbiah (1898), Ponnambalam et al. (2001) and Seifi et al. (2000) have employed the application of Kumaraswamy distribution in hydrology and related areas.

Bayesian estimation of the Kumaraswamy distribution has not been fully discussed in detail. The unknown parameter β is known as a random variable from a given probability distribution, with the prior information about the value of the parameter of interest β to the observed data. This research work focused on obtaining the Bayesian estimates of the shape parameters β of the Kumaraswamy distribution under different loss functions and different prior distributions.

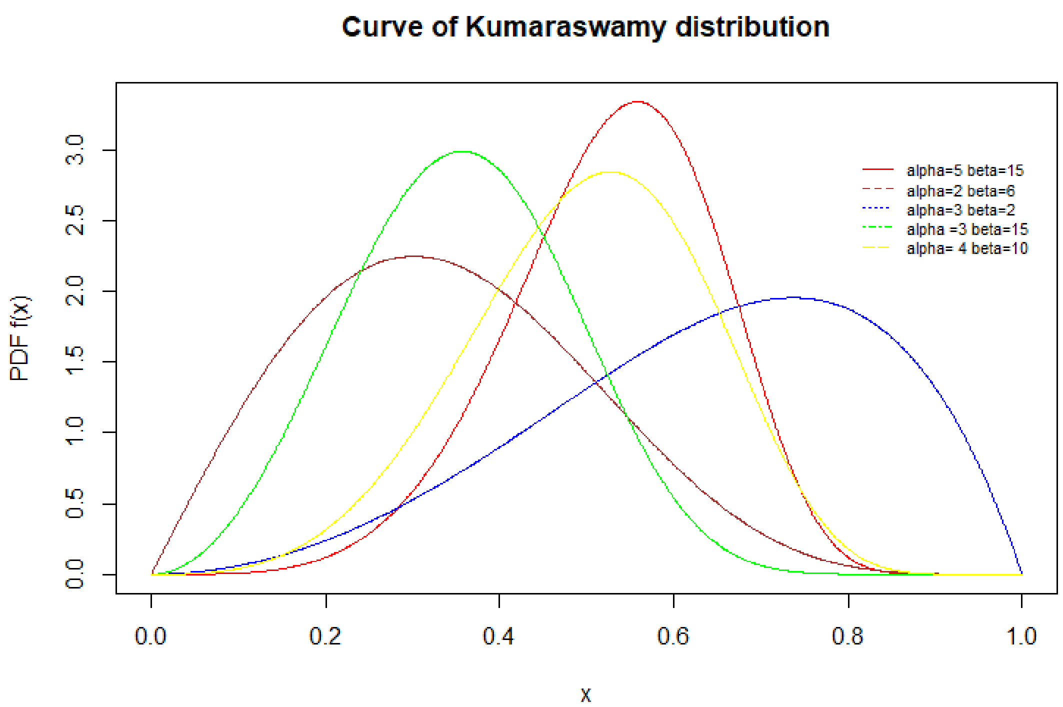

Figure 1: Probability Distribution Function curve of the Kumaraswamy under different parameters combinations.

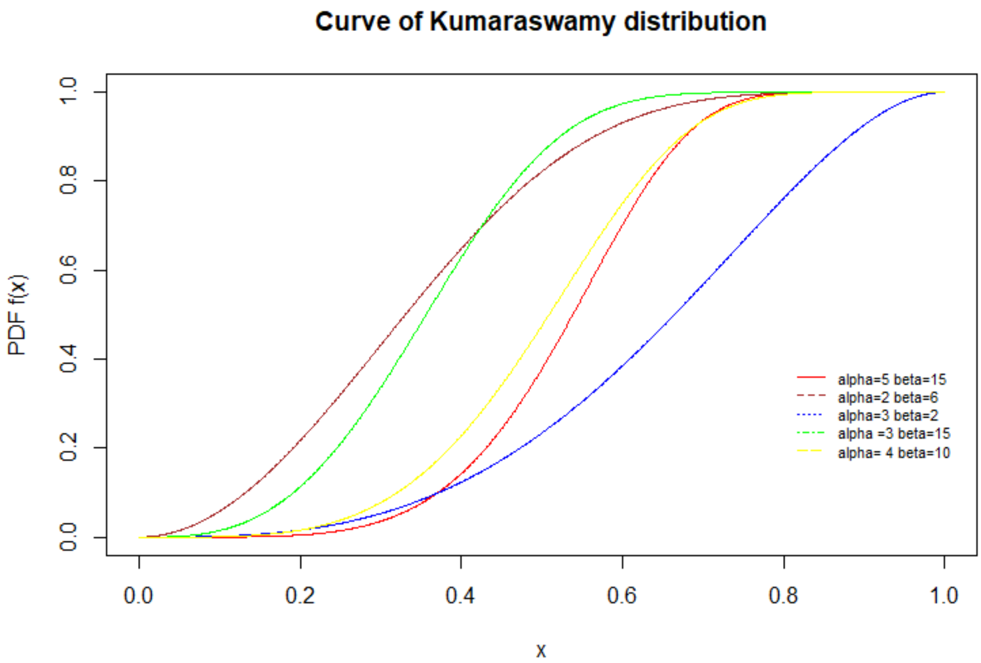

Figure 2: Cumulative Distribution Function Curve of the Kumaraswamy under different parameters combination.

2.0 The Maximum Likelihood Estimation

Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a Kumaraswamy distribution in (1). Therefore the likelihood function of (1) is given by

$$L(x|\beta) \propto \beta^n e^{-\beta T} \tag{4}$$

where $T = ln(1 - x_i^{\alpha})^{-1}$. The log-likelihood function of (1) can be expressed as

$$logL(x|\beta) = n\log\beta - \beta T \tag{5}$$

solving $\frac{\partial[logL(x|\beta)]}{\partial\beta} = 0$, we obtain the Maximum Likelihood Estimator (MLE) $\hat{\beta}$ for the parameter β as

$$\hat{\beta} = \left(\frac{T}{n}\right)^{-1} \tag{6}$$

2.1 Bayes' Theorem

The Bayes' theorem can be expressed as the conditional distribution of β given x is given as

$$h_i(\beta|x) = \frac{f(x|\beta)g_i(\beta)}{\int f(x|\beta)g_i(\beta)\,\partial\beta} \tag{7}$$

where $f(x|\beta)$ is the likelihood function of the distribution, $g(\beta)$ is the prior probability distribution for the parameter β and $h(\beta|x)$ is the posterior probability distribution.

2.2 Prior Information

In Bayesian inference, a prior probability distribution is often called the prior of an unknown parameter β which is used to express the uncertainty about the parameter β before the data is taken into consideration.

Although, at times priors are chosen according to one's subjective knowledge and belief that is why the Bayesian approach is sometimes called a subjective approach Dar et al. (2017). In this research work two types of prior distributions will be adopted namely; noninformative prior distribution (such as an extension of Jeffreys prior distribution, Hartigan Prior distribution and Uniform Prior distribution) and Informative prior distribution (such as Gamma Prior distribution and Inverted levy Prior distribution).

2.2.1 Prior Elicitation of Kumaraswamy Distribution with Extension of Jeffrey's Prior Distribution

Hudia and Huda(2014), proposed an extension of Jeffrey's prior distribution of the form

$$g(\beta) \propto \left[\sqrt{I(\beta)}\right]^{c_1} \tag{8}$$

where

$$I(\beta) = -nE\left(\frac{\partial^2 lnf(x|\beta)}{\partial\beta}\right) \tag{9}$$

From (1),

$$\begin{aligned} f(x|\beta) &\propto \beta\exp\{-\beta ln(1-x^\alpha)^{-1}\} \\ lnf(x|\beta) &\propto ln\beta - \beta ln(1-x^\alpha)^{-1} \\ \frac{\partial lnf(x,\beta)}{\partial\beta} &\propto \frac{1}{\beta} - ln(1-x^\alpha)^{-1} \\ \frac{\partial^2 lnf(x,\beta)}{\partial\beta^2} &\propto -\frac{1}{\beta^2} \end{aligned} \tag{10}$$

So, (9) becomes

$$I(\beta) = \frac{n}{\beta^2} \tag{11}$$

and the prior distribution for the Kumaraswamy distribution can be expressed as

$$g_1(\beta) \propto \left(\frac{1}{\beta}\right)^{c_1} \tag{12}$$

Remark 2.1: If $c_1 = 1$ in (12), we will have Jeffrey's prior. That is

$$g_{11}(\beta) \propto \frac{1}{\beta} \tag{13}$$

Remark 2.2: If $c_1 = \frac{3}{2}$, we will have a Hartigan or modified Jeffrey's prior. That is

$$g_{12}(\beta) \propto \left(\frac{1}{\beta}\right)^{3/2} \tag{14}$$

Remark 2.3: If $c_1 = 0$, we will have a Uniform prior $U\left(0, k = \frac{1}{p}\right)$. That is

$$g_{13}(\beta) \propto 1 \tag{15}$$

Thus, $g_{13} = p$ where p is the constant of proportionality.

2.2.2 Prior Elicitation of Kumaraswamy Distribution with Gamma Prior Distribution

The prior density distribution for the parameter β is assumed to follow a Gamma distribution with parameters a and b. The density function is therefore expressed as

$$g_2(\beta) = \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-b\beta} \tag{16}$$

2.2.3 Prior Elicitation of Kumaraswamy Distribution with Inverted Levy Prior Distribution

The prior density distribution for the parameter β is assumed to follow an Inverted Levy distribution of the form

$$g_3(\beta) = \sqrt{\frac{\phi}{2\pi}} \beta^{-\frac{1}{2}} e^{-\frac{\beta\phi}{2}} \tag{17}$$

2.3 The Loss Functions consideration for Bayesian Estimation

Loss function was introduced into statistics by Abraham Wald in the middle of 20^{th} Century. This requires the selection of an estimator. In this paper, we considered the use of squared error loss function, Al-Bayyati's loss function and Precautionary loss function for a better comparison of Bayes' estimators.

2.3.1 Squared Error Loss Function

The Squared Error Loss Function (SELF) is expressed by

$$l_{sq}(\hat{\beta}, \beta) = c(\hat{\beta} - \beta)^2 \tag{18}$$

The risk function for β under the SELF is

$$R_{RSQ}(\hat{\beta}) = \int_0^\infty c(\hat{\beta} - \beta)^2 h_i(\beta|x)\, \partial\beta \qquad i = 1,2,3 \tag{19}$$

2.3.2 Precautionary Loss Function (PLF)

The Precautionary Loss Function (PLF) is given as

$$l_{PLF}(\hat{\beta},\beta) = \frac{c(\hat{\beta}-\beta)^2}{\hat{\beta}} \tag{20}$$

The risk function for β under the PLF is

$$R_{RPLF}(\hat{\beta}) = \int_0^\infty \frac{c(\hat{\beta}-\beta)^2}{\hat{\beta}} h_i(\beta|x)\,\partial\beta \qquad i = 1,2,3 \tag{21}$$

2.3.3 Al-Bayatti's Loss Function (ALF)

The Al-Bayyati's Loss Function (ALF) is expressed by

$$l_{ALF}(\hat{\beta},\beta) = \beta^{c2}(\hat{\beta}-\beta)^2; \qquad c_2 \in \Re^+ \tag{22}$$

The risk function for β under the ALF is

$$R_{RALF}(\hat{\beta}) = \int_0^\infty \frac{c(\hat{\beta}-\beta)^2}{\hat{\beta}} h_i(\beta|x)\,\partial\beta \qquad i = 1,2,3 \tag{23}$$

where $\hat{\beta}$ is an estimate of the parameter β in all cases.

3.0 Bayesian Estimation of parameter β

3.1 Posterior Distribution of Kumaraswamy Distribution using the Extension of Jeffrey's Prior Distribution

Following (7), the posterior distribution can be expressed as

$$h_1(\beta|x) = \frac{\beta^n e^{-\beta T}\left(\frac{1}{\beta}\right)^{c_1}}{\int_0^\infty \beta^n e^{-\beta T}\left(\frac{1}{\beta}\right)^{c_1}}\partial\beta$$

$$h_1(\beta|x) = \frac{T^{n-c_1+1}}{\Gamma(n-c_1+1)}\beta^{n-c_1}e^{-\beta T} \tag{24}$$

which follows a Gamma distribution with parameters $G(n-c_1+1,T)$ with $Mean(x) = \frac{n-c_1+1}{T}$ and $Var(x) = \frac{n-c_1+1}{T^2}$.

3.1.1 Bayesian estimate of parameter β under the Squared-Error Loss Function using Extension of Jeffrey's Prior Distribution

Following (19), the risk function is written as

$$R_{(SQ,EJ)}(\hat{\beta}) = \int_0^\infty (\hat{\beta} - \beta)^2 \frac{T^{n-c_1+1}}{\Gamma(n-c_1+1)} \beta^{n-c_1} e^{-\beta T} \, \partial\beta$$

$$R_{(SQ,EJ)}(\hat{\beta}) = c\hat{\beta}^2 + \frac{c\Gamma(n-c_1+4)}{T^2\Gamma(n-c_1+1)} - \frac{2\hat{\beta}c\Gamma(n-c_1+2)}{T\Gamma(n-c_1+1)} \tag{25}$$

Now, on solving $\frac{\partial R_{(SQ,EJ)}}{\partial \hat{\beta}} = 0$, we obtained the Baye's estimator $\hat{\beta}$ as

$$\hat{\beta}_{(SQ,EJ)} = \frac{(n-c_1+1)}{T} \tag{26}$$

3.1.2 Bayesian estimate of parameter β under Al-Bayyati's Loss Function using Extension of Jeffrey's Prior Probability

Following (23), the risk function is written as

$$R_{(AL,EJ)}(\hat{\beta}) = \int_0^\infty \beta^{c_2} (\hat{\beta} - \beta)^2 \frac{T^{n-c_1+1}}{\Gamma(n-c_1+1)} \beta^{n-c_1} e^{-\beta T} \, \partial\beta$$

$$R_{(AL,EJ)}(\hat{\beta}) = \frac{\hat{\beta}^2\Gamma(n-c_1+c_2+1)}{T^{c_2}\Gamma(n-c_1+1)} - \frac{2\hat{\beta}\Gamma(n-c_1+c_2+2)}{\Gamma(n-c_1+1)T^{c_2}} + \frac{\Gamma(n-c_1+c_2+3)}{\Gamma(n-c_1+1)T^{c2+1}} \tag{27}$$

Now, on solving $\frac{\partial R_{(AL,EJ)}}{\partial \hat{\beta}} = 0$, we obtained the Baye's estimator $\hat{\beta}$ as

$$\hat{\beta}_{(AL,EJ)} = \frac{(n-c_1+c_2+1)}{T} \tag{28}$$

3.1.3 Bayesian estimate of parameter β under Precautionary Loss Function using Extension of Jeffrey's Prior Probability

Following (21), the risk function is written as

$$R_{(PL,EJ)}(\hat{\beta}) = \int_0^\infty \frac{c(\hat{\beta}-\beta)^2}{\hat{\beta}} \frac{T^{n-c_1+1}}{\Gamma(n-c_1+1)} \beta^{n-c_1} e^{-\beta T} \, \partial\beta$$

$$R_{(PL,EJ)}(\hat{\beta}) = c\hat{\beta} + \frac{c\Gamma(n-c_1+3)}{\hat{\beta}T^2\Gamma(n-c_1+1)} - \frac{2c\Gamma(n-c_1+2)}{T\Gamma(n-c_1+1)} \tag{29}$$

Now, on solving $\frac{\partial R_{(PL,EJ)}}{\partial \hat{\beta}} = 0$, we obtained the Baye's estimator $\hat{\beta}$ as

$$\hat{\beta}_{(PL,EJ)} = \frac{\sqrt{(n-c_1+2)(n-c_1+1)}}{T} \tag{30}$$

3.2 Posterior Distribution of Kumaraswamy Distribution using the Gamma Prior Distribution.

Following (7), the posterior distribution can be expressed as

$$h_2(\beta|x) = \frac{\beta^n e^{-\beta T} \beta^{a-1} e^{-b\beta}}{\int_0^\infty \beta^n e^{-\beta T} \beta^{a-1} e^{-b\beta}} \partial\beta$$

$$h_2(\beta|x) = \frac{(T+b)^{n+a}}{\Gamma(n+a)} \beta^{n+a-1} e^{-\beta(T-b)} \tag{31}$$

which follows an Gamma distribution with parameters $G(n+a, T+b)$ with $Mean(x) = \frac{n+a}{T+b}$ and $Var(x) = \frac{n+a}{(T+b)^2}$

3.2.1 Bayesian estimate of parameter β under Squared-Error Loss Function using Gamma Prior Probability

Following (19), the risk function is written as

$$R_{(SQ,GP)}(\hat{\beta}) = \int_0^\infty (\hat{\beta} - \beta)^2 \frac{(T+b)^{n+a}}{\Gamma(n+a)} \beta^{n+a-1} e^{-\beta(T-b)} \partial\beta$$

$$R_{(SQ,GP)}(\hat{\beta}) = c\hat{\beta}^2 + \frac{c\Gamma(n+a+2)}{(T+b)^2\Gamma(n+a)} - \frac{2\hat{\beta}c\Gamma(n+a+1)}{(T+b)\Gamma(n+b)} \tag{32}$$

Now, on solving $\frac{\partial R_{(SQ,GP)}}{\partial\hat{\beta}} = 0$, we obtained the Baye's estimator $\hat{\beta}$ as

$$\hat{\beta}_{(SQ,GP)} = \frac{(n+a)}{(T+b)} \tag{33}$$

3.2.2 Bayesian estimate of parameter β under Al-Bayyati's Loss Function using Gamma Prior Probability

Following (23), the risk function is written as

$$R_{(AL,GP)}(\hat{\beta}) = \int_0^\infty \beta^{c_2} (\hat{\beta} - \beta)^2 \frac{(T+b)^{n+a}}{\Gamma(n+a)} \beta^{n+a-1} e^{-\beta(T-b)} \partial\beta$$

$$R_{(AL,GP)}(\hat{\beta}) = \frac{\hat{\beta}^2\Gamma(n+a+c_2)}{(T+b)^{c_2}\Gamma(n+b)} - \frac{2\hat{\beta}\Gamma(n+a+c_2+1)}{\Gamma(n+a)(T+b)^{c_2+1}} + \frac{\Gamma(n+a+c_2+2)}{\Gamma(n+a)(T+b)^{c2+2}} \tag{34}$$

Now, on solving $\frac{\partial R_{(AL,GP)}}{\partial\hat{\beta}} = 0$, we obtained the Baye's estimator $\hat{\beta}$ as

$$\hat{\beta}_{(AL,GP)} = \frac{(n+a+c_2)}{(T+b)} \tag{35}$$

3.2.3 Bayesian estimate of parameter β under Precautionary Loss Function using Gamma Prior Probability

Following (21), the risk function is written as

$$R_{(PL,GP)}(\hat{\beta}) = \int_0^\infty \frac{c(\hat{\beta}-\beta)^2}{\hat{\beta}} \frac{(T+b)^{n+a}}{\Gamma(n+a)} \beta^{n+a-1} e^{-\beta(T+b)}\, \partial\beta$$

$$R_{(PL,GP)}(\hat{\beta}) = c\hat{\beta}^2 + \frac{c\Gamma(n+a+2)}{\hat{\beta}(T+b)^2\Gamma(n+a)} - \frac{2c\Gamma(n+a+1)}{(T+b)\Gamma(n+b)} \tag{36}$$

Now, on solving $\frac{\partial R_{(PL,GP)}}{\partial\hat{\beta}} = 0$, we obtained the Baye's estimator $\hat{\beta}$ as

$$\hat{\beta}_{(PL,EJ)} = \frac{\sqrt{(n+a+1)(n+a)}}{(T+b)} \tag{37}$$

3.3 Posterior Distribution of Kumaraswamy Distribution using the Inverted levy Prior Distribution

Following (7), the posterior distribution can be expressed as

$$h_3(\beta|x) = \frac{\beta^n e^{-\beta T} \beta^{\frac{1}{2}} e^{-\frac{\beta\phi}{2}}}{\int_0^\infty \beta^n e^{-\beta T} \beta^{\frac{1}{2}} e^{-\frac{\beta\phi}{2}} d\beta}$$

$$h_3(\beta|x) = \frac{(T+\frac{\phi}{2})^{n+\frac{1}{2}}}{\Gamma(n+\frac{1}{2})} \beta^{n-\frac{1}{2}} e^{-\beta(T+\frac{\phi}{2})} \tag{38}$$

which follows a Gamma distribution with parameters $G(n+\frac{1}{2}, T+\frac{\phi}{2})$ with $Mean(x) = \frac{n+\frac{1}{2}}{\frac{\phi}{2}}$ and $Var(x) = \frac{n+\frac{1}{2}}{(\frac{\phi}{2})^2}$

3.3.1 Bayes' estimation under Squared-Error Loss Function using Inverted levy Prior Distribution

Following (19), the risk function is written as

$$R_{(SQ,IL)}(\hat{\beta}) = \int_0^\infty (\hat{\beta}-\beta)^2 \frac{(T+\frac{\phi}{2})^{n+\frac{1}{2}}}{\Gamma(n+\frac{1}{2})} \beta^{n-\frac{1}{2}} e^{-\beta(T+\frac{\phi}{2})} d\beta$$

$$R_{(SQ,IL)}(\hat{\beta}) = c\hat{\beta}^2 + \frac{c\Gamma(n+\frac{5}{2})}{(T+\frac{1}{2})^2\Gamma(T+\frac{\phi}{2})} - \frac{2\hat{\beta}c\Gamma(n+\frac{3}{2})}{(T+\frac{\phi}{2})\Gamma(n+\frac{1}{2})} \tag{39}$$

Now, on solving $\frac{\partial R_{(SQ,IL)}}{\partial\hat{\beta}} = 0$, we obtained the Baye's estimator $\hat{\beta}$ as

$$\hat{\beta}_{(SQ,IL)} = \frac{(n+\frac{1}{2})}{(T+\frac{\phi}{2})} \tag{40}$$

3.3.2 Bayes' estimation under Al-Bayyati's Loss Function using Inverted levy Prior Distribution

Following (23), the risk function is written as

$$R_{(AL,IL)}(\hat{\beta}) = \int_0^\infty \beta^{c_2}(\hat{\beta}-\beta)^2 \frac{(T+\frac{\phi}{2})n+\frac{1}{2}}{\Gamma(n+\frac{1}{2})}\beta^{n-\frac{1}{2}}e^{-\beta(T+\frac{\phi}{2})}d\beta$$

$$R_{(AL,IL)}(\hat{\beta}) = \frac{\hat{\beta}^2\Gamma(n+\frac{1}{2})}{(T+\frac{\phi}{2})^{c_2+1}\Gamma(n+\frac{1}{2})} - \frac{2\hat{\beta}\Gamma(n+c_2+\frac{3}{2})}{\Gamma(n+\frac{1}{2})(T+\frac{\phi}{2})^{c_2+1}} + \frac{\Gamma(n+c_2+\frac{5}{2})}{\Gamma(n+\frac{1}{2})(T-\frac{\phi}{2})^{c2+2}} \quad (41)$$

Now, on solving $\frac{\partial R_{(AL,IL)}}{\partial\hat{\beta}} = 0$, we obtained the Baye's estimator $\hat{\beta}$ as

$$\hat{\beta}_{(AL,EJ)} = \frac{(n+c_2+\frac{1}{2})}{(T+\frac{\phi}{2})} \quad (42)$$

3.3.3 Bayes' estimation under Precautionary Loss Function using Inverted levy Prior Distribution

Following (21), the risk function is given as

$$R_{(PL,IL)}(\hat{\beta}) = \int_0^\infty \frac{c(\hat{\beta}-\beta)^2}{\hat{\beta}} \frac{(T+\frac{\phi}{2})n+\frac{1}{2}}{\Gamma(n+\frac{1}{2})}\beta^{n-\frac{1}{2}}e^{-\beta(T+\frac{\phi}{2})}d\beta$$

$$R_{(PL,IL)}(\hat{\beta}) = c\hat{\beta} + \frac{c\Gamma(n+\frac{5}{2})}{\hat{\beta}(T+\frac{\phi}{2})^2\Gamma(n+\frac{1}{2})} - \frac{2c\Gamma(n+\frac{3}{2})}{(T+\frac{\phi}{2})\Gamma(n+\frac{1}{2})} \quad (43)$$

Now, on solving $\frac{\partial R_{(PL,IL)}}{\partial\hat{\beta}} = 0$, we obtained the Baye's estimator $\hat{\beta}$ as

$$\hat{\beta}_{(PL,IL)} = \frac{\sqrt{\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right)}}{\left(T+\frac{\phi}{2}\right)} \quad (44)$$

4.0 Monte-Carlo Study and Results

In this reseach work, samples where drawn from a Kumaraswamy distribution with parameters $\alpha = 0.5$, $\beta = 0.5$, 1 and 1.5 with a sample sizes *n* = 100, 200, 500. The obtained results were repeated 10,000 times. Tables 1-3 displayed the estimates of parameter β of the Kumaraswamy distribution and the associated Mean Square Errors (MSEs) under each of the loss functions for different prior distributions. We set $c_2 = 2$, $\phi = 1$ and 3, *a* = 1 and 3, *b* = 1 and 3 and $c_1 = 0,\ 1$ and $\frac{3}{2}$.

Table 1: The Bayes (posterior) estimates of β and MSE under the Squared Error Loss Function

n	β	Jeffrey's Prior	Hartigan Prior	Uniform Prior	Gamma Prior		Inverted levy Prior	
		$c_1 = 1$	$c_1 = \frac{3}{2}$	$c_1 = 0$	$a = b = 1$	$a = b = 1$	$\phi = 1$	$\phi = 3$
100	0.5	0.5057	0.5030	0.5027	0.4976	0.4821	0.5058	0.5035
		(0.0027)	(0.0026)	(0.0027)	(0.0025)	(0.0023)	(0.0026)	(0.0025)
	1.0	1.0114	1.0061	1.0041	0.9893	0.9515	1.0088	1.0085
		(0.0104)	(0.0103)	(0.0102)	(0.0096)	(0.0086)	(0.0106)	(0.0102)
	1.5	1.5171	1.5056	1.5063	1.4781	1.4040	1.5118	1.4889
		(0.0235)	(0.0231)	(0.0232)	(0.0217)	(0.0186)	(0.0233)	(0.0216)
200	0.5	0.5002	0.5013	0.5043	0.4985	0.4914	0.5032	0.5021
		(0.0013)	(0.0013)	(0.0013)	(0.0012)	(0.001)	(0.0013)	(0.0012)
	1.0	1.0044	1.0130	1.0112	0.9932	0.9762	1.0050	1.0005
		(0.0051)	(0.0051)	(0.0052)	(0.0050)	(0.0048)	(0.0051)	(0.0049)
	1.5	1.5085	1.5030	1.5160	1.4890	1.4512	1.5050	1.4942
		(0.0116)	(0.0111)	(0.0118)	(0.0111)	(0.0100)	(0.0114)	(0.0111)
500	0.5	0.5009	0.5001	0.5023	0.4997	0.4966	0.5012	0.5004
		(0.0005)	(0.0005)	(0.0005)	(0.0005)	(0.0005)	(0.0004)	(0.0005)
	1.0	1.0019	1.001	1.0036	0.9979	0.9893	1.0013	0.9995
		(0.0020)	(0.0020)	(0.0020)	(0.0019)	(0.0019)	(0.0020)	(0.002)
	1.5	1.5041	1.5017	1.5049	1.4958	1.4802	1.5032	1.4981
		(0.0046)	(0.0045)	(0.0046)	(0.0045)	(0.0043)	(0.0045)	(0.0046)

Table 2: The Bayes (posterior) estimates of β and MSE under the Al-Bayyati's Loss Function when $c_2 = 2$

n	β	Jeffrey's Prior	Hartigan Prior	Uniform Prior	Gamma Prior		Inverted levy Prior	
		$c_1 = 1$	$c_1 = \frac{3}{2}$	$c_1 = 0$	$a = b = 1$	$a = b = 1$	$\phi = 1$	$\phi = 3$
100	0.5	0.5158	0.5131	0.5128	0.5173	0.5218	0.5158	0.5135
		(0.0028)	(0.0027)	(0.0027)	(0.0028)	(0.0027)	(0.0027)	(0.0026)
	1.0	1.0316	1.0263	1.0243	1.0293	1.0299	1.0289	1.0286
		(0.0108)	(0.0108)	(0.0106)	(0.0104)	(0.0101)	(0.0110)	(0.0106)
	1.5	1.5474	1.5359	1.5366	1.5378	1.5198	1.5418	1.5185
		(0.0244)	(0.0241)	(0.0242)	(0.0235)	(0.0218)	(0.0243)	(0.0224)
200	0.5	0.5073	0.5063	0.5935	0.5085	0.5114	0.5082	0.5071
		(0.0013)	(0.0013)	(0.0013)	(0.0013)	(0.0013)	(0.0052)	(0.0013)
	1.0	1.0145	1.0130	1.0213	1.0132	1.0158	1.0150	1.0105
		(0.0051)	(0.0051)	(0.0053)	(0.0052)	(0.0053)	(0.0052)	(0.0050)
	1.5	1.5236	1.5181	1.5311	1.5189	1.5101	1.5200	1.5091
		(0.0118)	(0.0114)	(0.0119)	(0.0011)	(0.0108)	(0.0116)	(0.0114)
500	0.5	0.5029	0.5014	0.5043	0.5037	0.5046	0.5032	0.5023
		(0.0005)	(0.0005)	(0.0005)	(0.0005)	(0.0005)	(0.0005)	(0.0005)
	1.0	1.0059	1.0050	1.0077	1.0059	1.0052	1.0053	1.0035
		(0.0020)	(0.0020)	(0.0020)	(0.0020)	(0.0020)	(0.0020)	(0.0020)
	1.5	1.5041	1.5077	1.5109	1.5033	1.5040	1.5092	1.5041
		(0.0046)	(0.0045)	(0.0046)	(0.0045)	(0.0045)	(0.0045)	(0.0046)

Table 3: The Bayes (posterior) estimates of β and MSE under the Precautionary Loss Function

n	$\boldsymbol{\beta}$	**Jeffrey's Prior**	**Hartigan Prior**	**Uniform Prior**	**Gamma Prior**		**Inverted levy Prior**	
		$c_1 = 1$	$c_1 = \frac{3}{2}$	$c_1 = 0$	$a = b = 1$	$a = b = 1$	$\phi = 1$	$\phi = 3$
100	0.5	0.5808	0.5055	0.5052	0.5098	0.5144	0.5083	0.5060
		(0.0027)	(0.0026)	(0.0026)	(0.0027)	(0.0026)	(0.0026)	(0.0025)
	1.0	1.0165	1.0112	1.0091	1.0143	1.0152	1.0139	1.0135
		(0.0105)	(0.0104)	(0.0103)	(0.0101)	(0.0097)	(0.0107)	(0.0103)
	1.5	1.5246	1.5131	1.5118	1.5154	1.4981	1.5193	1.4963
		(0.0237)	(0.0234)	(0.0235)	(0.0228)	(0.0212)	(0.0235)	(0.0218)
200	0.5	0.5035	0.5026	0.5068	0.5048	0.5077	0.50443	0.5034
		(0.0013)	(0.0013)	(0.0013)	(0.0013)	(0.0012)	(0.0013)	(0.0012)
	1.0	1.0070	1.0055	1.0106	1.0057	1.0083	1.0075	1.0030
		(0.0050)	(0.0050)	(0.0052)	(0.0052)	(0.0052)	(0.0051)	(0.0049)
	1.5	1.5123	1.5068	1.5177	1.5077	1.4990	1.5088	1.4980
		(0.0116)	(0.0112)	(0.0114)	(0.0114)	(0.0106)	(0.0114)	(0.0112)
500	0.5	0.5014	0.5014	0.50278	0.5022	0.5031	0.5017	0.5008
		(0.0005)	(0.0005)	(0.0005)	(0.0005)	(0.0005)	(0.0005)	(0.0005)
	1.0	1.0029	1.0020	1.0046	1.0029	1.0023	1.0023	1.0005
		(0.0020)	(0.0020)	(0.00200)	(0.0020)	(0.0020)	(0.0020)	(0.0020)
	1.5	1.5056	1.5032	1.5064	1.5033	1.4996	1.5047	1.4996
		(0.0046)	(0.0045)	(0.0046)	(0.0045)	(0.0045)	(0.00451)	(0.0046)

6.0 Discussion

The various estimates of the shape parameter β of the Kumaraswamy distribution provided by the Bayesian estimators at five different chosen priors over varying sample sizes are reported in Tables 1 to 3. For each parameter estimate reported, its corresponding MSE was determined as provided in the parenthesis.

All the five Bayesian estimators considered performed creditably well by providing good estimates of the shape parameter β of the Kumaraswamy distribution at different chosen sample sizes under the three loss functions considered. However, further assessment of the performances of these estimators showed that all these estimators yielded the best results under the square error loss function.

Without loss of generality, it can be observed from various results in Tables 1 to 3 that the performances of all the Bayesian estimators improved tremendously as the sample size increases irrespective of the type of loss function adopted. In fact, it is quite obvious that all the five Bayesian estimators yielded similar results under the three loss functions considered at large sample sizes.

7.0 Conclusions

This research work examined the efficiency of Bayesian estimators of the shape parameter of the Kumaraswamy distribution under different prior considerations and at three chosen loss functions.

Various results showed that all the prior distributions assumed on the parameter were quite good under the three loss functions considered. However, it was observed that the performances of the Bayesian estimators were in their best form under the square error loss function under which the MSEs of all the estimators were relatively smaller especially at small sample sizes.

Generally, it can be observed that the Bayesian estimates under the informative prior distributions proved to be better than those provided under the noninformative prior distributions. Specifically, the results obtained under the Inverted levy prior distribution were quite more efficient than others, especially under the SELF.

It can be concluded that the Bayesian estimators of the shape parameter of Kumaraswamy distribution under the five chosen priors provided good estimates of the parameter at all the three loss functions considered.

However, the results provided under the Inverted levy prior (an informative prior) using the SELF appear slightly better than those reported under the noninformative priors. It can, therefore, be recommended that Bayesian estimation of the shape parameter of the Kumaraswamy distribution is better considered an informative prior for the associated gains in relative efficiency.

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