Transmuted Kumaraswamy-Inverse Exponential Distribution and Its Properties

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Abstract---**This article introduces a four-parameter probability distribution obtained by introducing an additional parameter to the Kumaraswamy-Inverse Exponential distribution. The three-parameter Kumaraswamy-Inverse Exponential distribution was generalized to its four-parameter variante entitled Transmuted Kumaraswamy-Inverse Exponential Distribution (TK-IED) using the quadratic rank transmutation map introduced by [5]. Mathematical expressions for its moments, moment generating function (mgf) and the limiting behavior of the proposed model were presented. The parameters of the new distribution were estimated using the method of maximum likelihood.**

Keywords- *Kumaraswamy-Inverse exponential; Moments; Moment generating function*

I. INTRODUCTION

The process of parameter(s) induction to a given baseline distribution has attracted the attention of so many researchers in recent years. The induction of one or more additional shape parameter(s) to a baseline probability model makes it more flexible especially in terms of studying its tail characteristics. Inducing parameter(s) to an existing distribution(s) is a welcome idea for obtaining more flexible new families of distributions. A new technique of inducing a parameter to any baseline distribution purposely to make it more flexible and capable of capturing more tail properties was introduced in [5]. The research used the quadratic rank transmutation map (QRTM) in order to develop a flexible family of distributions. Some special distributions that can be obtained by the Transmuted family of the distributions were highlighted and discussed in [1].A random variable X has the transmuted-G family if the pdf and cdf are define through the QRTM method by;

$$F(x) = F(x;\xi,\eta) = G(x;\xi)[1+\eta-\eta G(x;\xi)] \quad (1)$$

$$f(x) = f(x;\xi,\eta) = g(x;\xi)[1+\eta-2\eta G(x;\xi)] \quad (2)$$

Where $G(x,\xi)$ is the baseline cdf and $g(x,\xi)$, is the baseline pdf, η is a transmutation parameter taking allowable values: $-1 \le \eta \le 1$

Now, to obtain the CDF and pdf of the Transmuted Kumaraswamy-Inverse Exponential Distribution denoted as (TK-IED), we used equation (1) and (2) where

$$G(x;\xi) = G(x;(a,b,\lambda) = \left\{1-\left(1-e^{-\frac{a\lambda}{x}}\right)^{b}\right\} \quad (3)$$

$$g(x;\xi) = g(x;(a,b,\lambda) = \frac{ab\lambda}{x^2}e^{-\frac{a\lambda}{x}}\left(1-e^{-\frac{a\lambda}{x}}\right)^{b-1} \quad (4)$$

Cordeiro and de Castro [2] studied a new class of the Kumaraswamy generalized distributions (denoted by the Kw–G distribution) based on the Kumaraswamy distribution.

Oguntunde *et al.* [3] identified the statistical properties such as hazard function, survival function, moment and moment generating function of Kumaraswamy inverse Exponential Distribution.

Oguntunde and Adejumo [4] studied a two parameter probability model called Transmuted Inverse Exponential Distribution using a QRTM approach. The study derived its statistical properties and highlighted its usefulness in modeling datasets related to breast and bladder cancers.

The aim of this paper is to derive some of the statistical properties of the TK-IED.

II. RESEARCH METHODOLOGY

A. The cumulative distribution function and probability density function of TK-IED.

$$F(x;a,b,\lambda,\eta) = \left\{1-\left(1-e^{-\frac{a\lambda}{x}}\right)^{b}\right\}\left\{1+\eta-\eta\left(1-\left(1-e^{-\frac{a\lambda}{x}}\right)^{b}\right)\right\} \quad (5)$$

$$f(x;a,b,\lambda,\eta) = \frac{ab\lambda}{x^2}e^{-\frac{a\lambda}{x}}\left(1-e^{-\frac{a\lambda}{x}}\right)^{b-1}\left\{1+\eta-2\eta\left(1-\left(1-e^{-\frac{a\lambda}{x}}\right)^{b}\right)\right\} \quad (6)$$

Equation (6) can be further reduced to:

$$f(x;a,b,\lambda,\eta)=\frac{ab\lambda}{x^2}e^{-\frac{a\lambda}{x}}\left(1-e^{-\frac{a\lambda}{x}}\right)^{b-1}\left\{1-\eta+2\eta\left(1-\left(1-e^{-\frac{a\lambda}{x}}\right)^{b}\right)\right\} \quad (7)$$

where $a, b,, \lambda > 0\ and : -1 \le \eta \le 1$

$$f(x;a,b,\lambda,\eta)=\frac{ab\lambda(1-\eta)}{x^2}e^{-\frac{a\lambda}{x}}\left(1-e^{-\frac{a\lambda}{x}}\right)^{b-1}+\frac{2ab\lambda\eta}{x^2}e^{-\frac{a\lambda}{x}}\left(1-e^{-\frac{a\lambda}{x}}\right)^{2b-1} \quad (8)$$

Now using binomial series expansion, we have

$$\left(1-e^{-\frac{a\lambda}{x}}\right)^{b-1}=\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}e^{-\frac{a\lambda j}{x}} \quad (9)$$

$$\left(1-e^{-\frac{a\lambda}{x}}\right)^{2b-1}=\sum_{j=0}^{\infty}(-1)^j\binom{2b-1}{j}e^{-\frac{a\lambda j}{x}} \quad (10)$$

Hence by utilizing equation (9) and (10), we rewrite (8) as

$$f(x;a,b,\lambda,\eta)=\frac{ab\lambda(1-\eta)}{x^2}e^{-\frac{a\lambda}{x}}\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}e^{-\frac{a\lambda j}{x}}+\frac{2ab\lambda\eta}{x^2}e^{-\frac{a\lambda}{x}}\sum_{j=0}^{\infty}(-1)^j\binom{2b-1}{j}e^{-\frac{a\lambda j}{x}} \quad (11)$$

By choosing some appropriate values for parameters: a=a,b=b,λ=c and η=d, we provide a possible shape for the CDF and pdf of the TK-IE distribution as shown in Figures 1 and 2:

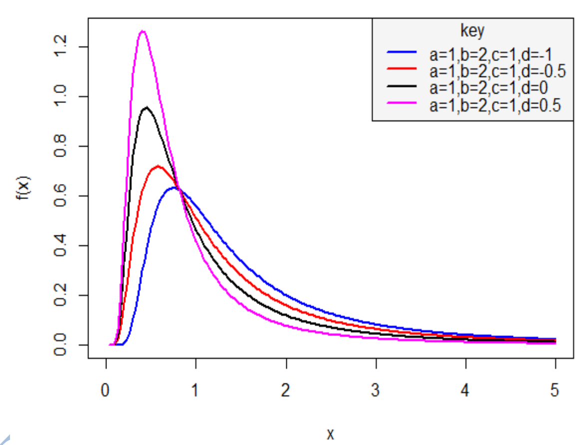

Figure 1: pdf plot of TK-IED

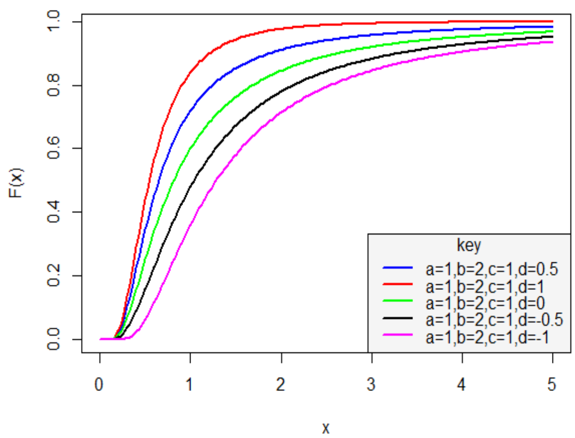

Figure 2: CDF plot of TK-IED

B. Moment

The r^{th} moment of a TK-IED is given by;

$$\mu_r'=(a\lambda)^r\Gamma(1-r)\sum_{j=0}^{\infty}(-1)^j\frac{(1-j)^{r-1}}{j!}\gamma_j \quad (12)$$

Proof: the r^{th}moment of a random variable X with pdf is define by;

$$\mu_r'=E(x^r)=\int_{-\infty}^{\infty}x^r f(x)dx \quad (13)$$

$$\mu_r'=\int_{-\infty}^{\infty}x^r\left\{\left(\frac{ab\lambda(1-\eta)}{x^2}e^{-\frac{a\lambda}{x}}\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}e^{-\frac{a\lambda j}{x}}+\frac{2ab\lambda\eta}{x^2}e^{-\frac{a\lambda}{x}}\sum_{j=0}^{\infty}(-1)^j\binom{2b-1}{j}e^{-\frac{a\lambda j}{x}}\right)\right\}dx \quad (14)$$

$$\mu_r'=ab\lambda(1-\eta)\int_0^{\infty}x^{r-2}e^{-\frac{a\lambda}{x}}\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}e^{-\frac{a\lambda j}{x}}dx+2ab\lambda\eta\int_0^{\infty}x^{r-2}e^{-\frac{a\lambda}{x}}\sum_{j=0}^{\infty}(-1)^j\binom{2b-1}{j}e^{-\frac{a\lambda j}{x}}dx \quad (15)$$

$$\mu_r'=ab\lambda(1-\eta)\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\int_0^{\infty}x^{r-2}e^{-\frac{a\lambda(1+j)}{x}}dx+2ab\lambda\eta\sum_{j=0}^{\infty}(-1)^j\binom{2b-1}{j}\int_0^{\infty}x^{r-2}e^{-\frac{a\lambda(1+j)}{x}}dx \quad (16)$$

But,

$$\int_0^{\infty}x^{r-2}e^{-\frac{a\lambda(1+j)}{x}}dx=(a\lambda)^r(1+j)^{r-1}\Gamma(1-r) \quad (17)$$

By replacing (17) to (16) we have,

$$\mu_r'=ab\lambda(1-\eta)\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}(a\lambda)^{r-1}(1+j)^{r-1}\Gamma(1-r)+2ab\lambda\eta\sum_{j=0}^{\infty}(-1)^j\binom{2b-1}{j}(a\lambda)^{r-1}(1+j)^{r-1}\Gamma(1-r) \quad (18)$$

Equation (18) can be reduced to

$$\mu_r' = ab\lambda(1-\eta)\sum_{j=0}^{\infty}(-1)^j \frac{(b-1)!}{j!\,(b-j-1)!}(a\lambda)^{r-1}(1+j)^{r-1}\Gamma(1-r) + 2ab\lambda\eta$$
$\sum_{j=0}^{\infty}(-1)^j \frac{(2b-1)!}{j!(2b-j-1)!}(a\lambda)^{r-1}(1+j)^{r-1}\Gamma(1-r)$
(19)

$$\mu_r' = ab\lambda(1-\eta)\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma b}{j!\,\Gamma(b-j)}(a\lambda)^{r-1}(1+j)^{r-1}\Gamma(1-r) + 2ab\lambda\eta$$
$\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma 2b}{j!\Gamma(2b-j)}(a\lambda)^{r-1}(1+j)^{r-1}\Gamma(1-r)$

(20)$\mu_r' = (a\lambda)^r\Gamma(1-r)\sum_{j=0}^{\infty}(-1)^j \frac{(1+j)^{r-1}}{j}\left(\frac{(1-\eta)\Gamma(b+1)}{(\Gamma(b-j))} + \frac{\eta\Gamma(2b+1)}{\Gamma(2b-j)}\right)$

Where $\gamma_j = \left(\frac{(1-\eta)\Gamma(b+1)}{(\Gamma(b-j))} + \frac{\eta\Gamma(2b+1)}{\Gamma(2b-j)}\right)$

$$\mu_r' = (a\lambda)^r\Gamma(1-r)\sum_{j=0}^{\infty}(-1)^j \frac{((1+j)^{r-1}}{j}\gamma_j$$

C. Moment Generating Function

The moment generating function (mgf) of a random variable X is given as:

$M_x(t) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^j \frac{(at\lambda)^m(1+j)^{m-1}\Gamma(1-m)}{j!m!}\gamma_j$(21)

Proof: $M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx\,dx$ (22)

$M_x(t) = \int_0^{\infty} e^{tx}\left\{\frac{ab\lambda(1-\eta)}{x^2}e^{-\frac{a\lambda}{x}}\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}e^{-\frac{a\lambda j}{x}} + \frac{2ab\lambda\eta}{x^2}e^{-\frac{a\lambda}{x}}\sum_{j=0}^{\infty}(-1)^j\binom{2b-1}{j}e^{-\frac{a\lambda j}{x}}\right\}dx$ (23)

$= ab\lambda(1-\eta)\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\int_0^{\infty} e^{tx}\frac{1}{x^2}e^{\frac{a\lambda(1+j)}{x}}dx + 2ab\lambda\eta\sum_{j=0}^{\infty}(-1)^j\binom{2b-1}{j}\int_0^{\infty} e^{tx}\frac{1}{x^2}e^{-\frac{a\lambda(1+j)}{x}}dx$ (24)

But according to Maclaurin's series expansion

$e^{tx} = \sum_{m=0}^{\infty}\frac{t^m x^m}{m!}$ (25)

By substituting (25) in (24)

$$= ab\lambda(1-\eta)\sum_{j=0}^{\infty}(-1)^j\frac{(b-1)!}{j!\,(b-j-1)!}\sum_{m=0}^{\infty}\frac{t^m}{m!}\int_0^{\infty}x^{m-2}e^{-\frac{a\lambda(1+j)}{x}}dx + 2ab\lambda\eta\sum_{j=0}^{\infty}(-1)^j\frac{(2b-1)!}{j!\,(2b-j-1)!}\sum_{m=0}^{\infty}\frac{t^m}{m!}\int_0^{\infty}x^{m-2}e^{-\frac{a\lambda(1+j)}{x}}dx$$
(26)

But,

$\int_0^{\infty} x^{m-2}e^{-\frac{a\lambda(1+j)}{x}}dx = (a\lambda)^m(1+j)^{m-1}\Gamma(1-m)$(27)

Proof:

Let $y = \frac{a\lambda(1+j)}{x}$, as $x \to 0, y \to \infty$ and $x \to \infty, y \to 0$

So, $x = \frac{a\lambda(1+j)}{y}, \frac{dx}{dy} = \frac{-a\lambda(1+j)}{y^2}$

$$-\int_0^{\infty}(a\lambda)^{m-2}(1+j)^{m-2}\left(\frac{1}{y}\right)^{m-2}\left(\frac{-a\lambda(1+j)}{y^2}\right)e^{-y}dy$$

$$(a\lambda)^{m-1}(1+j)^{m-1}\int_0^{\infty}y^{-m+2-2}e^{-y}dy$$
$$(a\lambda)^{m-1}(1+j)^{m-1}\int_0^{\infty}y^{(1-m)-1}e^{-y}dy$$
$$(a\lambda)^{m-1}(1+j)^{m-1}\Gamma(1-m)$$

Hence,

$$\int_0^{\infty}x^{m-2}e^{-\frac{a\lambda(1+j)}{x}}dx = (a\lambda)^m(1+j)^{m-1}\Gamma(1-m)$$

By replacing (27) in (26) we have;

$M_x(t) =$
$\sum_{m=0}^{\infty}\frac{(at\lambda)^m\Gamma(1-m)}{m!}\left((1-\eta)\Gamma(b+1)\sum_{j=0}^{\infty}\frac{(-1)^j(1+j)^{m-1}}{j!\Gamma(b-j)} + \eta\Gamma(2b+1)\sum_{j=o}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j(1+j)^{m-1}}{j!\Gamma(2b-j)}\right)$

(28)

$$M_x(t) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^j\frac{(at\lambda)^m(1+j)^{m-1}\Gamma(1-m)}{j!\,m!}\gamma_j$$

D. Limiting behavior

Here, we seek to investigate the asymptotic behavior of the TK-IED in equation (6) as $x \to 0$ and as $x \to \infty$.

$\lim_{x\to 0} f(x;a,b,\lambda,\eta) = \lim_{x\to 0}\left(\frac{ab\lambda}{x^2}e^{-\frac{a\lambda}{x}}\left(1-e^{-\frac{a\lambda}{x}}\right)^{b-1}\left\{1+\eta-2\eta\left(1-\left(1-e^{-\frac{a\lambda}{x}}\right)^b\right)\right\}\right) = 0$(29)

And

$\lim_{x\to 0} f(x;a,b,\lambda,\eta) = \lim_{x\to\infty}\left(\frac{ab\lambda}{x^2}e^{-\frac{a\lambda}{x}}\left(1-e^{-\frac{a\lambda}{x}}\right)^{b-1}\left\{1+\eta-2\eta\left(1-\left(1-e^{-\frac{a\lambda}{x}}\right)^b\right)\right\}\right) = 0$

(30)

Equation (29) and (30) shows that the proposed model has at least one mode.

III. Estimation of Parameters of the TK-IED

The estimation of the parameters of the TK-IED is done by using the method of maximum likelihood estimation. Let $x_1, x_2, \ldots, x_n$, be a random sample from the TK-IED with unknown parameters. The total log-likelihood function for θ is obtained from f(x) as follows:

$\ln L(\theta) = n\ln a + n\ln a + n\ln b + n\ln\lambda - 2\sum_{i=0}^{n}\ln x_i - \sum_{i=0}^{n}\frac{a\lambda}{x_i} + (b-1)\sum_{i=0}^{n}\ln(1-e^{-\frac{a\lambda}{x_i}}) + \sum_{i=0}^{n}\ln\left\{1+\eta-2\eta\left(1-\left(1-e^{-\frac{a\lambda}{x}}\right)^b\right)\right\}$ (31)

So, differentiating l(θ) partially with respect to each of the parameter (a, b, λ, η) and setting the results equal to zero gives the maximum likelihood estimates of the respective parameters. The partial derivatives of l(θ) with respect to each parameter or the score function is given by:

$$\frac{\delta L}{\delta a} = \frac{n}{a} - \lambda \sum_{i=0}^{n} \frac{1}{x_i} + (b-1) \sum_{i=0}^{n} \frac{\lambda}{x_i\left(e^{\frac{a\lambda}{x}} - 1\right)}$$

$$+ \sum_{i=0}^{n} \frac{2ab\eta e^{-\frac{a\lambda}{x}}\left(1-e^{-\frac{a\lambda}{x}}\right)^{b-1}}{x_i\left(1+\eta-2\eta\left(1-\left(1-e^{-\frac{a\lambda}{x}}\right)^{b}\right)\right)} = 0 \quad (32)$$

$$\frac{\delta L}{\delta b} = \frac{n}{b} + \sum_{i=0}^{n} \ln\left(1 - e^{-\frac{a\lambda}{x}}\right)$$

$$+2\eta \sum_{i=0}^{n} \frac{\left(\ln\left(1-e^{-\frac{a\lambda}{x}}\right)\right)\left(1-e^{-\frac{a\lambda}{x}}\right)}{x_i\left(1+\eta-2\eta\left(1-\left(1-e^{-\frac{a\lambda}{x}}\right)^{b}\right)\right)} = 0 \quad (33)$$

$$\frac{\delta L}{\delta \lambda} = \frac{\delta L}{\delta a} = \frac{n}{a} - \lambda \sum_{i=0}^{n} \frac{1}{x_i} + (b-1) \sum_{i=0}^{n} \frac{a}{x_i\left(e^{\frac{a\lambda}{x}} - 1\right)}$$

$$+ \sum_{i=0}^{n} \frac{2b\lambda\eta e^{-\frac{a\lambda}{x}}\left(1-e^{-\frac{a\lambda}{x}}\right)^{b-1}}{x_i\left(1+\eta-2\eta\left(1-\left(1-e^{-\frac{a\lambda}{x}}\right)^{b}\right)\right)} = 0 \quad (34)$$

$$\frac{\delta L}{\delta \eta} = \sum_{i=0}^{n} \frac{1-2\left(1-e^{-\frac{a\lambda}{x}}\right)^{b}}{\left(1+\eta-2\eta\left(1-\left(1-e^{-\frac{a\lambda}{x}}\right)^{b}\right)\right)} = 0$$

(35)

Hence, the MLE is obtained by solving this nonlinear system of equations. Solving this system of nonlinear equations is complicated, we can therefore use statistical software to solve the equations numerically.

IV. Conclusion

We have successfully defined a four parameter distribution called Transmuted Kumaraswamy Inverse Exponential Distribution (TK-IED). The distribution is positively skewed and its shape is unimodal. The r^{th} moment and moment generating function were derived. We observed that the moment for the TK-IED only exist for $r < 1$. Therefore, the first moment, second moment and other higher-order moments does not exist.

Acknowledgment

The authors would like to thank the referees for their valuable comments and suggestions which greatly improved the paper.

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